both the E–E case and the E–L case [which was initialized with four parcels/cell and using the optimum parcel adaptivity parameters of (4, 9, Δh_b)]. Both cases used an LOR of 5 and yielded good correlation with experimental results, however, the E–E case exhibited better agreement and its density contours were found to be less noisy. The E–E case also required roughly a factor of two less CPU time (145 vs 337 min) and memory (1.3 vs 2.1 Mwords) on a Cray Y-MP than the E–L case.[9] Computational requirements were estimated for a hypothetical case that used the E–L technique also with an LOR of 5 but without adaptive parcels and with enough initial parcels to ensure one per cell at the highest refinement level. The resulting E–L case without parcel adaptivity required far more CPU time and memory (2900 min and 50.8 Mwords) than its equivalent parcel-adaptive E–L counterpart. This shows the strong potential for the parcel adaptivity scheme, especially as one progresses to three-dimensional flows. From these comparisons and results of Sivier et al.,[1,9] however, the E–E technique is the most computationally efficient for shock attenuation of this particular flowfield.

IV. Conclusions

A novel Lagrangian parcel-adaptive method has been developed and added to a two-phase compressible flow solver to allow a more accurate and more efficient study of particle and droplet flows. The flow computations employ the FEM–FCT scheme, while the finite difference Lagrangian parcel equations are solved directly for each parcel. The performance of the scheme under different implementations was evaluated using a test case of an unsteady shock attenuation in a dusty gas. The parcel adaptivity allowed an order of magnitude savings in computational resources as compared to a nonadaptive parcel treatment and was most effective when cell structured initialization was used along with an optimum refinement distribution length scale and optimum limits on the parcels per cell. The adaptive Lagrangian treatment of particles computational characteristics, however, was not as efficient as when the particle phase was treated in an Eulerian manner.

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Transonic Equivalence Rule Involving Lift and Shocks

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Introduction

FOR aerodynamic configurations with sufficiently high lift, the transonic equivalence rule[1–4] must be modified for both linear and nonlinear lift contributions.[5–11] The latter analyses show that for a thin wing having swept leading edges, with smooth lift and thickness distributions, the outer-flow region has a nonlinear structure determined principally by an equivalent line source and a line doublet. Apart from serving as a helpful guide to conceptual aerodynamic design, the equivalence rule and its extension can be useful in transonic wind-tunnel wall corrections, as well as studying sonic-boom impact at low-supersonic flight speed.

The present article re-examines the inner and outer solutions and their matching and clarifies the accuracy levels of the theory with consideration of the higher order corrections and shock discontinuities in the outer flow. The matching involving shocks in the outer flow (to be distinguished from a shock imbedded and terminated within the inner region), as shown by Cole and Malmuth[12] and Malmuth[13] for the axisymmetric case, leads to requirements on the local shock geometry and its location that may, in turn, determine their admissibility. Certain classes of admissible solution behavior and the open issues are noted.

Inner Solution in the Outer Limit: Corrections

The perturbation velocity potential ϕ in the inner region not far from the wing/body cross section $[r/b = \mathcal{O}(1)]$ is assumed to be expandable in the small parameter $\epsilon = [(\gamma+1)M_\infty^2 \tau\lambda^3]^{1/2}$, as (cf. Ref. 5)

$$\phi/\alpha U b = \varphi_I + \epsilon\varphi_{II} + \epsilon^2\varphi_{III} + \epsilon^3\varphi_{IV} \tag{1}$$

where $\lambda \equiv b/l$, α is an effective attack angle, τ is a characteristic cross-section thickness ratio, b is a reference spanwise length scale, and l is a reference streamwise length scale. For the present purpose, only the first two terms are of concern, which can be decomposed into five parts φ_0, φ_1, φ_2, ψ_2, and ψ_2' as[5]

$$\varphi_I = \varphi_0, \qquad \varphi_{II} = \sigma_1^{-1}\varphi_1 + \tfrac{1}{8}\sigma_1\left(\varphi_2 + \Gamma\psi_2 + \Gamma^2\psi_2'\right) \tag{2}$$

with

$$\sigma_1 \equiv (\gamma+1)^{\frac{1}{2}} M_\infty \lambda^{\frac{3}{2}} \alpha/\tau^{\frac{1}{2}}, \qquad \Gamma \equiv 8(\gamma+1)^{-1}\lambda^{-2} \tag{3}$$

The results for φ_0, φ_1, φ_2, ψ_2, and ψ_2' presented in Eqs. (3.18), (3.19), (3.21), (3.22), and (3.24) of Ref. 5 need corrections for transcribing and printing errors. The corrected inner solution ϕ for large r is written in the outer variable $\eta = \epsilon r$, omitting term of order $\epsilon^2\eta^{-2}$, $\epsilon^2\eta^{-1}$, $\epsilon^2 \ell n\, \epsilon$, ϵ^2, η as

$$\phi/\tau U b \equiv \Phi \sim (2\pi)^{-1} S_e'(x) \ell n\, \eta + (2\pi)^{-1}\sigma_1 F(x)\eta^{-1}\sin\omega + \beta_0(x) + \epsilon(2\pi)^{-1}\left[\tilde{D}_1(x)\eta^{-1}\cos\omega + \sigma_1 m_{32}\eta^{-2}\sin 2\omega\right] + \Phi_{\text{non}} \tag{4}$$

with

$$\sigma_*^2 = \sigma_1^2 |\ell n\, \epsilon| \tag{5}$$

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$$S_e(x) = S_c(x) + \frac{\sigma_*^2}{8\pi}\left[F_x^2 + \frac{\pi}{|\ln\epsilon|}\left(\int_{a_1}^{a_2}\chi(x, y_1)\,\mathrm{d}y_1 + \Gamma E(x)\right)\right]$$
$$= S_c(x) + \left(\sigma_*^2/8\pi\right)\left(1 + \frac{1}{2}|\ln\epsilon|^{-1}\right)F_x^2$$
$$+ \left(\sigma_*^2/8\right)|\ln\epsilon|^{-1}[4T(x) + \Gamma E(x)] \tag{6}$$

$$\tilde{D}_1 = -\bar{y}\frac{\mathrm{d}}{\mathrm{d}x}S_c(x) - \frac{\sigma_1^2}{8}\left[\frac{1}{\pi}|\ln\epsilon|(F_x m_{32})_x\left(1 - \frac{1}{4}|\ln\epsilon|^{-1}\right)\right.$$
$$+ \frac{\mathrm{d}}{\mathrm{d}x}\int_{a_1}^{a_2}\chi(x, y_1)y_1\,\mathrm{d}y_1 - \Gamma\int_{a_1}^{a_2} y_1[\varphi_0(x, y_1)]\frac{\partial^2}{\partial x^2}Z_0\,\mathrm{d}y_1$$
$$\left. - (\gamma + 1)\Gamma^2\int_{a_1}^{a_2}[\varphi_0(x \cdot y_1)]_y Z_0(x, y_1)\,\mathrm{d}y_1\right] \tag{7}$$

$$\Phi_{\mathrm{non}} = \left(\sigma_1^2/64\pi^2\right)\left\{\left(F_x^2\right)_x[2(\ln\eta)^2 + \cos 2\omega]\right.$$
$$\left. - \epsilon(F_x m_{32})_x\eta^{-1}[4\ln\eta\cos\omega - \cos 3\omega]\right\} \tag{8}$$

$$\beta_0(x) = b_1(x) + \tfrac{1}{8}\sigma_*^2[b_2(x) + \Gamma|\ln\epsilon|^{-1}\tilde{b}_2(x)] \tag{9}$$

$$\chi(x, y) = -\frac{1}{\pi}[\varphi_0(x, y)]_x$$
$$\times\,\mathrm{PV}\int_{a_1}^{a_2}[\varphi_0(x, y_1)]_x\left[\frac{y}{y_1 - y} + \ln|y_1 - y|\right]\mathrm{d}y_1 \tag{10}$$

where $[\,]$ signifies a jump across the lifting surface discontinuity, F_x, $(m_{32})_x$, S_c, $\bar{y}$, and Z_0 are the normalized sectional lift, rolling moment, geometrical cross-sectional area, spanwise centroid location and camber ordinate defined in Eq. (3.20) of Ref. 5; the functions $E(x)$ and $T(x)$ can be calculated from Eqs. (3.23) or (5.5) and (5.4), respectively, therein; the function $\chi(x)$ is defined after Eq. (3.21). Functions χ and $(Z_0)_{xx}$ in Eq. (7) vanish on the trailing vortex sheet. Following Ref. 5, x has been normalized by the characteristic chordwise length scale l.

Among the terms in Eq. (4) are those of an equivalent line source of strength $S_e'(x)$, a line doublet associated with the lift $\sigma_1 F(x)$, a weak line quadrupole $\epsilon\sigma_1 m_{32}$, and also a weak line doublet resulting from bilateral asymetry $\tilde{D}_1(x)$. The group denoted by Φ_{non} are the anharmonic terms resulting from the nonlinear inhomogeneity of the crossflow equation.

Apart from changing $(1 - \frac{1}{2}|\ln\epsilon|^{-1})$ and $y_1[\varphi_0(x, y_1)]$ to $(1 - \frac{1}{4}|\ln\epsilon|^{-1})$ and $-\Gamma y_1[\varphi_0(x, y_1)]$, respectively, in the original expression for $\tilde{D}_1$, the result differs from the corresponding expressions of Ref. 5, Eq. (4.8) therein, in the restoration to $\tilde{D}_1$ the last term inside the square bracket in Eq. (7) and to Φ_{non} the order-ϵ corrections in Eq. (8) that were inadvertently deleted from the expressions obtained for φ_2 and ψ_2'.

Outer Solution in the Inner Limit

In the notations of Ref. 5, the corresponding inner expansion of the outer solution, with $i = 2, 3$, also $l_2(\omega) = \cos\omega$, and $l_3(\omega) = \sin\omega$, is

$$\Phi \sim D^{(j)}(x)l_i(\omega)\eta^{-1} + \epsilon C_{ij}(x)[2l_i(\omega)l_j(\omega) - \delta_{ij}]\eta^{-2}$$
$$+ C_0(x)\ln\eta + C_1(x) + \frac{\partial}{\partial x}\left\{\frac{1}{8}D_x^{(j)}D_x^{(j)}\ln^2\eta\right.$$
$$\left. - \frac{1}{16}\left[D_x^{(2)}D_x^{(2)} - D_x^{(3)}D_x^{(3)}\right]\cos 2\omega - \frac{1}{8}D_x^{(2)}D_x^{(3)}\sin 2\omega\right\}$$
$$- \frac{\epsilon}{16\eta}\frac{\partial}{\partial x}\left\{\left(D_x^{(2)}G_x^{(2)} - D_x^{(3)}G_x^{(3)}\right)\cos 3\omega\right.$$
$$- \left(D_x^{(2)}G_x^{(3)} + D_x^{(3)}G_x^{(2)}\right)\sin 3\omega + 4\ln\eta\left[D_x^{(j)}D_x^{(j)}\cos\omega\right.$$
$$\left.\left. - \left(D_x^{(3)}G_x^{(2)} - D_x^{(2)}G_x^{(3)}\right)\sin\omega\right]\right\} \tag{11}$$

where, for convenience, $G^{(2)} = C_{22} - C_{33}$, $G^3 = C_{23}$. This expression, with remainders of order ϵ^2 and η deleted, is the same as Eq. (4.7) in Ref. 5, except for the last group of terms with the factor $\epsilon/16\eta$ that was deleted and is essential to completing the match with the inner solution.

The expressions for the lift-dominated case $\sigma_* \gg 1$ can likewise be obtained and matched [cf. Eq. (4.12) in Ref. 5]; they are identifiable with Eqs. (4) and (11) after rescaling ϕ and η with new scale factors $\alpha^2\lambda^3(\gamma + 1)M_\infty^2|\ln\epsilon'|bU$ and ϵ'/ϵ [cf. Eqs. (4.2a) and (4.2b) of Ref. 5].

Matching and the Equivalence Rule

The inner and outer solutions may now be matched, i.e., matching Eqs. (4) and (11) in the overlapping η range $\epsilon \ll \eta \ll 1$. In the presence of shocks reaching down from the outer flow, the following is still valid except in the shock vicinities, where the singularity strengths C_0, $D^{(2)}$, $D^{(3)}$ etc., in Eq. (11) must be readjusted across the shock to satisfy the shock conditions to be addressed later. The matching determines strength of line source $C_0(x)$, line doublets $D^{(2)}(x)$ and $D^{(3)}(x)$, and the line quadrupole C_{ij} of the outer (transonic) flow as

$$C_0(x) = (2\pi)^{-1}S_e'(x), \qquad D^{(2)}(x) = \epsilon(2\pi)^{-1}\tilde{D}_1(x)$$
$$D^{(3)}(x) = (2\pi)^{-1}\sigma_1 F(x) \tag{12}$$
$$C_{ij} = 0 \qquad \text{except} \qquad C_{32} = (2\pi)^{-1}\sigma_1 m_{32}(x)$$

The matching also identifies $\beta_0(x)$ with $C_1(x)$. Having matched the multipole terms in both solutions, the matching of Φ_{non} in Eq. (4), i.e., Φ_{non} of Eq. (8) with the remaining anharmonic terms of Eq. (11) also follows. Note in this regard that $G^{(2)} = C_{22} - C_{33} = 0$ and $D^{(2)} = \mathcal{O}(\epsilon)$, therefore, the last group of anharmonic terms of Eq. (11) is simply

$$-(\epsilon/64\pi^2)\sigma_1^2(F_x m_{32})_x(4\ln\eta\cos\omega - \cos 3\omega)/\eta$$

which matches the terms with factor $\epsilon(F_x m_{32})_x\eta^{-1}$ in Φ_{non} of Eq. (8) after omitting terms of the order $\epsilon^2\eta^{-1}$. With the corrected inner and outer expansions, the match in question has been established over $\epsilon \ll \eta \ll 1$ for fixed finite σ_* and $\Gamma/|\ln\epsilon|$ subject to errors (mismatches) of orders represented by

$$\epsilon^2\eta^{-2}, \qquad \epsilon^2\eta^{-1}, \qquad \epsilon^2\ln\epsilon, \qquad \epsilon^2, \qquad \text{and} \qquad \eta$$

It follows that the outer solution governed by the transonic small-disturbance equation

$$-\left(K\Phi_x + \tfrac{1}{2}\Phi_x^2\right)_x + \eta^{-1}(\eta\Phi_\eta)_\eta + \eta^{-2}\Phi_{\omega\omega} = 0 \tag{13}$$

with $K \equiv (M_\infty^2 - 1)/(\gamma + 1)M_\infty^2\tau\lambda$ and a uniform far field, i.e.,

$$\Phi \to 0, \qquad \text{as} \quad x^2 + \eta^2 \to \infty, \qquad \eta^2 \neq \mathcal{O}(1) \tag{14}$$

is determined by the source strength $S_e'(x)$, the doublet strength $\sigma_1 F(x)$, and to a lesser extent by the quadrupole strength $\epsilon\sigma_1 m_{32}(x)$, as well as another doublet strength $\epsilon\tilde{D}_1(x)$, that specify the Φ behavior in the inner limit $\eta \to 0$. The error of this outer solution at $\eta \neq 0$ that is determined up to the order ϵ is expected to be no worse than the order $\epsilon^2\ln\epsilon$. The function $C_1(x)$ furnishes a feedback in pressure from the nonlinear outer flow and remains unknown until the boundary-value problem is solved. The outer flow determined includes the shock jumps as discontinuities of the weak solution to the partial differential equation (13) subject to the thermodynamic requirement on the entropy increase.

Assuming that the flow in the outer region is completely determined by the four line-singularity strengths indicated, the nonlinear mixed-flow structures at $\eta \neq 0$ over geometrically different configurations must then be the same when expressed in the variables $(x/l, \epsilon r/b, \omega)$ for a fixed K, as long as the axial distributions

$$\frac{\mathrm{d}}{\mathrm{d}x}S_e, \qquad \sigma_1 F, \qquad \epsilon\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(\frac{\mathrm{d}}{\mathrm{d}x}F\right)\cdot\frac{\mathrm{d}}{\mathrm{d}x}(m_{32})\right], \qquad \text{and} \qquad \epsilon\tilde{D}_1$$

are the same, subject to (relative) error of the order $\epsilon^2 \ln \epsilon$, irrespective of detail spanwise and chordwise lift and thickness distributions, as well as the lack of bilateral symmetry. The flow quantities correlated must be scaled accordingly as

$$\phi/\tau U b, \qquad c_p/\tau\lambda, \qquad (M^2-1)/\epsilon^2, \qquad \text{etc.}$$

In this sense, these correlated configurations are equivalent. The corrections shown earlier in this Note affect the line doublet strength $\epsilon \tilde{D}_1$; hence they improve the rule's (algebraic) validity to the order $\epsilon^2 \ln \epsilon$ for the bilaterally asymmetrical case. The equivalence rule in Sec. 5 of Ref. 5 addresses specifically the simplier version, requiring only the specification of the equivalent cross-sectional area $S_e(x)$ and the axial lift up to x, $\sigma_1 F(x)$, for which the accuracy level cannot be maintained to the next order ϵ, unless the system has bilateral symmetry ($m_{32} = \tilde{D}_1 = 0$). As in Ref. 5, the equivalence rule so stated does not require the invariance of parameters such as σ_* and Γ_*, nor the gas specific heat ratio γ; it, thus, possesses more degrees of freedom for aerodynamic design than in the use of transonic similarity law.

Admissibility of Shock Jumps in Matching

We examine the question: How should the matching of Eqs. (4) and (11), hence the equivalence rule, be modified in the event that a discontinuity occurs and reaches the axis ($\eta \to 0$)? The prediction of the shock-root location appears to be possible, as has been suggested by Cole and Malmuth[12] and Malmuth[13] for the axisymmetric case.

The shock in the matching zone ($\epsilon \ll \eta \ll 1$) divides the flowfield in its vicinity into two distinct regions: an interior region where Eqs. (11) and (4) can be directly matched and an exterior region where the outer flow perceives the distributions of line source, line doublet, and line quadrupole distinctly different from those on the interior side, because of the shock discontinuity. In the following, these distributions from the inner solution will be readjusted across the shock before they are used to match with the outer flow solution on the exterior side, so that the jump requirements of the Rankine–Hugoniot relations are satisfied.

Now the equations governing the discontinuity surface $x = x^D(\eta, \omega)$, admissible to the weak solution of the partial differential equation (13) for the outer flow, are

$$[\Phi_x] : [\Phi_\eta] : [\Phi_\omega] = -1 : \frac{\partial x^D}{\partial \eta} : \frac{\partial x^D}{\partial \omega} \tag{15}$$

$$\langle K + \Phi_x \rangle = \left([\Phi_\eta]^2 + \eta^{-2}[\Phi_\omega]^2\right)/[\Phi_x]^2 \tag{16}$$

where $\langle\,\rangle$ denotes the arithmetical mean of values across the shock discontinuity surface.

We shall be concerned with determining those portions of the equivalent line-source and line-doublet distributions of the inner-flow solution to be matched by the outer solution on the exterior side that will complete the match over the shock vicinity not treated previously.

The outer expansion (in η) of the inner solution on the interior side is that given by Eq. (4), where line-doublet and line-source strengths $\sigma_1 F(x)$ and $S_e'(x)$, as well as other coefficients in the expansion, are prescribed by the thickness and lift distributions of the wing [cf. Eqs. (4–10)]. For simplicity, we omitted the line-quadruple contributions in Eqs. (4) and (8) that belong to an order ϵ higher. In order to include the axisymmetric case studied by Cole and Malmuth[12] and Malmuth[13] as a limit $\sigma_1 \to 0$ corresponding to a slender wing / body, we add to the right-hand side of Eq. (4) a term, $\Theta\eta^2(\ln\eta)^2$, where $\Theta = (\gamma+1)S_e''S_e'''/4$ with terms of order $|\ln \eta|^{-1}$ higher omitted. (Note S_e'' is identified with S' in Refs. 12 and 13.)

On the other, exterior side of the shock, one has a different set of doublet and source strength for the admissible expansion, that can be represented by $\sigma_1[F(x) + f(x)]$ and $[S_e' + q(x)]$ with corresponding changes of $(F_x^2)_x$ and $S_e''S_e'''$ to $\mathrm{d}(F_x + f_x)^2/\mathrm{d}x$ and $(S_e'+q)'(S_e'+q)''$, respectively. The function $\beta_0(x)$ in Eq. (4) must also allow a change to $[\beta_0(x) + b(x)]$ on the exterior side of the shock. The left- and right-hand members of Eq. (16) may accordingly be modified to allow the determination of f, q, b, etc., and $(x - x_0)$ at the shock $x = x^D(\eta, \omega)$, as well as the reference location x_0 itself. We seek results of $f(x)$, $q(x)$, and $b(x)$ for a small $(x - x_0)$, since a nonvanishing $(x - x_0)$ in the matching range $\epsilon \ll \eta \ll 1$ would imply an unbounded $\partial x^D/\partial\eta$ and $\langle\Phi_x\rangle$. Therefore, it suffices to consider the expansions in ascending powers in $(x - x_0)$

$$S_e'(x) = S_e'(x_0) + S_e''(x_0)(x-x_0) + \tfrac{1}{2}S_e'''(x_0)(x-x_0)^2 + \cdots \tag{17}$$

$$q(x) = q'(x_0)(x-x_0) + \tfrac{1}{2}q''(x_0)(x-x_0)^2 + \cdots \tag{18}$$

with similar expressions for $F(x)$, $f(x)$, $\beta_0(x)$, and $b(x)$. Substituting these truncated expansions into the modified Eq. (16) yields an algebraic equation of $(x - x_0)$ as a function of η and ω, for a vanishing η. The parametric dependence of $(x - x_0)$ on the derivatives of S_e, F, q, f, and b at $x = x_0$, as well as x_0 itself, may then be examined for its admissibility as a solution.

For the axisymmetric case ($F \equiv 0$, $S_e \equiv S_c$), the foregoing procedure applied to the limit $\sigma_1 \to 0$ yields a family of $(x^D - x_0)$ that vanishes with η, with undetermined q', b', or q'' at $x = x_0$. But the slopes of these solution, $\mathrm{d}x^D/\mathrm{d}\eta$, are logarithmically infinite as $\eta \to 0$, with one exception, namely,

$$x^D - x_0 \sim 2\pi (S_c''')_0^{-1}\Theta_0'\eta^2 \ln \eta \tag{19}$$

which can occur only at the location of maximum cross-sectional area, $S_c''(x_0) = 0$, determined completely by the body geometry. Except for a possible difference in the constant coefficient in Eq. (19), Cole and Malmuth's original result[12,13] is essentially recovered.

In the domain on a nonvanishing σ_1, the procedure leads to several families of $(x^D - x_0)$ vanishing with η, depending on whether $F_0' = 0$ or not. The case with a vanishing F' at $x = x_0$ gives solution families yielding logarithmically infinite $\partial x^D/\partial\eta$ in the limit $\eta \to 0$ and may be ruled out; whereas allowing a shock root of the outer-flow region to occur at a location where $F'(x_0) \neq 0$ leads to several interesting possibilities all having finite, nonvanishing slope as $\eta \to 0$. Exceptional among these is one occurring at $S_e''(x_0) = 0$ (under a nonvanishing F_0'). This yields

$$x - x_0 \sim \left(\beta_0'' + \tfrac{1}{2}b''\right)_0 \eta^2 \sin^2\omega \tag{20}$$

and, thus, approaches a normal shock of zero curvature in the case.

Closing Remarks

More detailed discussion on the admissible shock solutions in the matching range $\epsilon \ll \eta \ll 1$ will be presented in a fuller article. The issues on predictablity of the shock location and its behavior can not be settled without a thorough investigation involving extensive computational and experimental studies, inasmuch as the problem's uniqueness and its dependence on the global solution are open.

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Eigenvector Derivatives of Structures with Rigid Body Modes

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Introduction

BECAUSE of the increasing importance of sensitivity analysis in structural dynamics, development of effective and efficient methods for computing eigenvector derivatives has been an active area of research during the last three decades as shown in the excellent review paper by Haftka and Adelman.[1] The earliest method for computing eigenvector derivatives, apparently due to Fox and Kapoor,[2] requires all of the modes of a system and is computationally expensive. To improve computational efficiency, Nelson[3] developed an effective method to calculate eigenvector derivatives of the rth mode by just using the modal parameters of that mode. Application of Nelson's method, however, is limited to the case where only few eigenvector derivatives and few design parameters are of interest. To reduce computational effort involved in a wide range of applications in which a large number of eigenvector derivatives with respect to a large number of design parameters are required, an improvement to the truncated modal summation representation of eigenvector derivatives was proposed by Wang[4] in which a mode-acceleration type approach was used to obtain a static solution to approximate the contribution due to unavailable higher modes. In Ref. 5, an implicit method was further proposed by assuming that the eigenvector derivatives are spanned by the truncated mode shapes together with a residual static mode. Recently, Liu et al.[6] improved the accuracy of eigenvector derivatives of Refs. 4 and 5 by introducing higher order correction terms. Applications of the methods presented in Refs. 4–6, however, are practically limited since they are based on the prerequisite that the stiffness matrix of the structure to be analyzed is nonsingular and, hence, possesses no rigid body modes.

Since many practical structures possess rigid body modes due to insufficient structural constraints and some structural components are specifically analyzed under free-free conditions before they are coupled to form structural assemblies, this Note presents a new development that generalizes the methods of Refs. 4–6 so that structures with rigid body modes can be analyzed. By defining and solving a similar eigenvalue problem of a structure with rigid body modes, eigenvector derivatives can be accurately computed by using lower computed modes and a modified flexibility matrix. A numerical example is given to demonstrate the practicality of the proposed method.

Existing Modal Superposition Methods

The matrix representation of vibration eigenvalue problem is

$$[K]\{\phi\}_r - \lambda_r[M]\{\phi\}_r = \{0\} \tag{1}$$

Differentiating with respect to design parameter p,

$$[[K]-\lambda_r[M]]\frac{\partial\{\phi\}_r}{\partial p}+\left[\frac{\partial[K]}{\partial p}-\lambda_r\frac{\partial[M]}{\partial p}-\frac{\partial\lambda_r}{\partial p}[M]\right]\{\phi\}_r = 0 \tag{2}$$

Assume that $\{\phi\}_r$ is mass normalized such that

$$\{\phi\}_r^T[M]\{\phi\}_r = 1 \tag{3}$$

Without any loss of generality, the rth eigenvector derivatives can be expressed as[7]

$$\frac{\partial\{\phi\}_r}{\partial p} = \sum_{s=1}^{N}\beta_{rs}\{\phi\}_s \tag{4}$$

Substituting Eq. (4) into Eq. (2) and premultiplying Eq. (2) by $\{\phi\}_s^T$, β_{rs} ($s \neq r$) becomes

$$\beta_{rs} = \frac{1}{\lambda_r-\lambda_s}\{\phi\}_s^T\left[\frac{\partial[K]}{\partial p}-\lambda_r\frac{\partial[M]}{\partial p}\right]\{\phi\}_r \qquad s\neq r \tag{5}$$

Upon differentiation of Eq. (3) and subsequent substitution of Eq. (4), β_{rr} can be obtained as

$$\beta_{rr} = -\frac{1}{2}\{\phi\}_r^T\frac{\partial[M]}{\partial p}\{\phi\}_r \qquad s = r \tag{6}$$

Therefore, the eigenvector derivatives of rth mode become

$$\frac{\partial\{\phi\}_r}{\partial p} = \sum_{\substack{s=1\\ s\neq r}}^{N}\frac{1}{\lambda_r-\lambda_s}\{\phi\}_s^T\left[\frac{\partial[K]}{\partial p}-\lambda_r\frac{\partial[M]}{\partial p}\right]\{\phi\}_r\{\phi\}_s - \frac{1}{2}\{\phi\}_r^T\frac{\partial[M]}{\partial p}\{\phi\}_r\{\phi\}_r \tag{7}$$

Equation (7) represents the modal method proposed by Fox and Kapoor[2] that requires the availability of all of the modes of the system.

To reduce the computational cost, an improved modal method that aims to derive the required eigenvector derivatives approximately by using the calculated lower modes and the known flexibility matrix was developed by Wang.[4,5] Suppose that only few of the lower modes (m modes) of interest have been computed, then Eq. (7) can be modified to become

$$\frac{\partial\{\phi\}_r}{\partial p} = \sum_{\substack{s=1\\ s\neq r}}^{m}\frac{\{\phi\}_s^T\{F\}_r}{\lambda_r-\lambda_s}\{\phi\}_s + \sum_{s=m+1}^{N}\frac{\{\phi\}_s^T\{F\}_r}{\lambda_r-\lambda_s}\{\phi\}_s - \frac{1}{2}\{\phi\}_r^T\frac{\partial[M]}{\partial p}\{\phi\}_r\{\phi\}_r \tag{8}$$

where $\{F\}_r \equiv [\partial[K]/\partial p - \lambda_r\partial[M]/\partial p]\{\phi\}_r$ and we have assumed that $r < m$. If the eigenvalues are numbered according to their magnitude in ascending order, then $\lambda_s - \lambda_r \approx \lambda_s$ for $s > r$ and Eq. (8) can be approximated as

$$\frac{\partial\{\phi\}_r}{\partial p} = \sum_{\substack{s=1\\ s\neq r}}^{m}\frac{\{\phi\}_s^T\{F\}_r}{\lambda_r-\lambda_s}\{\phi\}_s + \sum_{s=m+1}^{N}\frac{\{\phi\}_s^T\{F\}_r}{-\lambda_s}\{\phi\}_s - \frac{1}{2}\{\phi\}_r^T\frac{\partial[M]}{\partial p}\{\phi\}_r\{\phi\}_r \tag{9}$$

The flexibility matrix $[K]^{-1}$, if it exists, can be written as

$$[K]^{-1} = [\phi][\lceil\lambda\rfloor]^{-1}[\phi]^T = \sum_{s=1}^{N}\frac{\{\phi\}_s\{\phi\}_s^T}{\lambda_s} \tag{10}$$

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